Prime Fuzzy Submodules and Primary Fuzzy Submodules

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Abstract
In this paper our aim is to extend some notions of ordinary prime and primary submodules into fuzzy prime and fuzzy primary submodules. Also we introduce and study other properties of prime and primary fuzzy submodules. Several results on fuzzy prime and fuzzy primary submodules are proved.

Keywords
Fuzzy Module, Fuzzy Submodule, Fuzzy Prime Submodule, Fuzzy Primary Submodule

I. Introduction and Background

Throughout this paper by R mean a commutative ring with identity and M is an R-module. A crisp set is defined by dichotomize the individuals into two types - members and non members.

one observes distinction exists between the members and nonmembers of the class represented by the crisp set. Many collections and categories we commonly employ, however, do not exhibit this characteristic. Their boundaries seem vague, and the transition from member to nonmember appears gradual not abrupt. this called fuzzy set which introduces vagueness by eliminating the sharp boundary dividing members of the class from nonmembers.In our live the more situations are very often not crisp and deterministic and they can not be described precisely. Such situations are characterized by vagueness or imprecision can not be answered just in yes or no.

Lotfi A. Zadeh [1] in 1965 introduced the notion of a fuzzy set to describe vagueness mathematically in its very abstractness and to solve such problems he gave a certain grade of membership to each member of a given set. This in fact laid the basic of fuzzy set theory. Zadeh has defined a fuzzy set as a generalization of characteristic function of a set wherein the degree of membership of an element is more general than merely "yes" or "no". A fuzzy set can be defined mathematically by assigning to each possible individual in the universe of discourse a value representing its grade of membership in the fuzzy set. This grade corresponds to the degree to which that individual is compatible with the concept represented by the fuzzy set. The membership grades are very often represented by real number values ranging in the closed interval between 0 and 1. The notion of fuzzy set was applied in algebra as one of the first branches from among various branches of pure mathematics . The first paper on fuzzy groups was published by A. Rosenfeld[2] in 1971, in which the concepts of fuzzy subgroupoid and fuzzy subgroups were introduced. In 1979 J.M. Anthony and H. Sherwood[3].Under the new notion which called triangular norm function by redefined fuzzy subgroup some results of Rosenfeld is studied .

After a considerable period of time W. J. Liu[4] opened the way towards the development of fuzzy algebraic structures by introducing the notions of fuzzy normal subgroup, fuzzy subring and the product of fuzzy sets. Liu[5] introduced the notion of a fuzzy ideal of a ring. N. Kuroki[6] demonstrated the utility of the notion of the fuzzy set in the more general setting of semigroups. The concepts of fuzzy fields and fuzzy linear spaces were introduced by S. Nanda [7] Ever since A. Rosenfeld introduced fuzzy sets in the realm of group theory, many researchers have been involved in extending the notions of abstract algebra to the broader framework of fuzzy setting. J. N. Mordeson, D. S. Malik, M. M. Zahedi, M. Das, M. K Chakraborty, B. B. Makamba, V. Murali, A. K. Katsaras, D. B. Liu, M. Asaad, P.S. Das,N. P. Mukherjee, P. Bhattacharya, F. I. Sidky, M. A. Mishref, and M. Akhul, T. K. Mukherjee, M. K. Sen, V. N. Dixit, N. Ajmal, R. Kumar are a few among the others who contributed a lot to the theory of fuzzy algebraic structures. As a consequence, a number of concepts have been formulated and explored.

The concept of fuzzy modules and L-modules were introduced by Negoita and Ralescu [8] and Mashinchiand Zahedi [9] respectively. Subsequently they were further studied by Golan[10]Muganda[11], Pan[12-13-14-15], Zahedi and Ameri[16-17-18-19]. The notion of free fuzzy modules was introduced by Muganda[11] in 1993 as an extension of free modules in the fuzzy context. In I994 Zahedi and Ameri[19] introduced the concept of fuzzy exact sequences in the category of fuzzy modules and in 1995 they introduced the concepts of fuzzy projective and injective modules[16].

Throughout this work we shall use the notion of fuzzy set to generalize some useful results which have been presented in the ordinary case by a great staff of researchers. Also we shall introduce some conditions to become some theorem which has a necessary side only to two sides necessary and sufficient. Therefore this paper presents a most qualitative study in mathematics.

Note : Throughout this paper, we do not distinguish between the statement of prime fuzzy and fuzzy prime.

Definition 1.1[1]: Let M be a non-empty set and let I be the closed interval [0,1]. A fuzzy set μ in M (a fuzzy subset μ of M) is a function from M to I.

Definition 1.2[1]: A fuzzy set μ of a set M is called constant if $\mu(x) = t$ for all $x \in M$, where $t \in [0,1]$.

Definition 1.3[1]: Let μ be a fuzzy set in M. μis called an empty set denoted by $\varnothing$ if and only if $\mu(x) = 0$ for all $x \in M$.

Definition 1.4[20]: Let $x_t : M \longrightarrow [0,1]$ be a fuzzy set in M, where $x \in M$, $t \in [0,1]$, defined by:

$$x_t(y) = \begin{cases} t & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

for all $y \in M$, x_t is called a fuzzy singleton or fuzzy point in M. If $x = 0$and $t= 1$, then:

$$0_1(y) = \begin{cases} 1 & \text{if } y = 0 \\ 0 & \text{if } y \neq 0 \end{cases}$$

$x_t \subseteq \mu$ if and only if $x_t(y) \leq \mu(y)$, for all $y \in M$ and if $t > 0$, then $\mu(x) \geq t$. If x_t and y_s fuzzy singletons, then $x_t+y_s=(x+y)_l$ and $x_t \circ y_s=(x.y)_l$, where $l = \min\{t,s\}$

Definition1.5[2]: Let $(G,\cdot)$ be a group and μ be a fuzzy set in G then μ is called a fuzzy group in G if for each $x,y \in G$:

$\mu(x \cdot y) \geq \min \mu(x), \mu(y)\}$.

$\mu(x) = \mu(x^{-1})$

A fuzzy subgroup of a fuzzy group μ is a fuzzy group λ satisfing $\lambda(x) \le \mu(x) \quad \forall\, x \in G$.

Definition1.6[21]: Let $(R,+,\cdot)$ be a ring and μ be a fuzzy set in R, then μ is called fuzzy ring in a ring $(R,+,\cdot)$ if for each $x,y \in R$:
$\mu(x+y) \ge \min\{\mu(x), \mu(y)\}$.
$\mu(x) = \mu(x^{-1})$.
$\mu(x \cdot y) \ge \min\{\mu(x), \mu(y)\}$.
A fuzzy subring of a fuzzy ring μ is a fuzzy ring λ satisfing $\lambda(x) \le \mu(x) \quad \forall\, x \in R$

Definition 1.7[20][22]: A fuzzy set μ in a ring R is called a fuzzy left(right) ideal of the ring R if for each $x,y \in R$:
$\mu(x-y) \ge \min\{\mu(x), \mu(y)\}$.
$\mu(xy) \ge \mu(y) \quad [\mu(xy) \ge \mu(x)]$.
In [23] when R is a commutative, Bhambri.S.K., Kumar R. and Kumar P introduced the definition:

Definition 1.8[23]: A fuzzy set μ in a commutative ring R is called a fuzzy ideal of the ring R if for each $x, y \in R$:
$\mu(x-y) \ge \min\{\mu(x), \mu(y)\}$.
$\mu(xy) \ge \max\{\mu(x), \mu(y)\}$.

Definition 1.9 [24-25]: Let R be a ring and let M be a left R-module.A fuzzy set μ in M is called a fuzzy left R-module if for each $x, y \in M$ and $r \in R$:
$\mu(x-y) \ge \min\{\mu(x), \mu(y)\}$.
$\mu(rx) \ge \mu(x)$.
$\mu(0) = 1$ (0 the zero element of M).

Definition 1.10[26-27-28]: Let μ, λ be two fuzzy subsets of M, and let r be any element of R, define $\mu+\lambda$,$r\mu$ of M as follows: for each $x \in M$.
$(\mu+\lambda)(x) = \sup\{\min\{\mu(a)+\lambda(b)\} : x=a+b\}$ for each $a,b \in M$.

$$(r\mu)(x) = \begin{cases} \sup\{\mu(a) : x = ra\,, a \in M\} \\ 0 \qquad \text{otherwise} \end{cases}$$

Definition 1.11[29]: Let μ be a fuzzy subsets of a ring R, and λ be a fuzzy subset of an R- module M. For every $x \in M$
Define
$(\mu\lambda)(x) = \sup\{\inf\{\mu(r_1), \mu(r_2), \ldots \mu(r_k)\lambda(x_1), \lambda(x_2), \ldots, \lambda(x_k)\}\}$ where $r_i \in R$ and $x_i \in M$.

Proposition 1.1[29]: Let μ and λ be fuzzy ideals of a ring R and let ρ and θ be submodules of an R- module M. Then the followings are satisfied.
$\mu+\lambda$ is a fuzzy ideal of R,
$\rho+\theta$ is a fuzzy submodule of M,
$\mu\lambda$ is a fuzzy ideal of R,
$\mu\rho$ is a fuzzy submodule of M,
$(\mu+\lambda)\rho = \mu\rho+\lambda\rho$,
$(\mu\lambda)\rho = \mu(\lambda\rho)$.

Definition 1.12[1]: Let μ and λbe fuzzy submodules of A fuzzy module A of an R- module M. The residual of μ and λ denoted by $[\mu:\lambda]$ is the fuzzy subset of R defined by:
$[\mu:\lambda] = \sup\{t \in [0,1] : r_t\, \lambda \subseteq \mu\}$ for all $r \in R$ that is:
$[\mu:\lambda] = \{r_t : r_t\, \lambda \subseteq \mu\}$, r_t is a fuzzy singleton of R. If $\lambda=(x_k)$, then;
$[\mu:(x_k)\,] = \{r_t : r_t\, x_k \subseteq \mu\}$, r_t is a fuzzy singleton of R
Here we mention the equivalent definition and it is very clear to showing the equivalence between the two definitions.

Definition 1.13[30]: For $\mu,\upsilon \in I^M$ and $\alpha \in I^R$ define the residual quotient $[\mu:\upsilon] \in I^R$ and $[\mu:\alpha] \in I^M$ as follows :
$[\mu:\upsilon] = \cup\{\eta : \eta \in I^R, \eta.\upsilon \subseteq \mu\}$
$[\mu:\alpha] = \cup\{\sigma : \sigma \in I^M, \alpha.\sigma \subseteq \mu\}$

Theorem 1.1[30]: Let $\mu,\upsilon \in I^M$ and $\alpha \in I^R$. Then
$[\mu:\upsilon].\upsilon \subseteq \mu$.
$\alpha.[\mu:\alpha] \subseteq \mu$.
$\alpha.\upsilon \subseteq \mu \Leftrightarrow \alpha \subseteq [\mu:\upsilon] \Leftrightarrow \upsilon \subseteq [\mu:\alpha]$

Proposition 1.2 [1]: Let μ and λbe fuzzy submodules of a fuzzy module A of an R- module M, then $[\mu:\lambda]$ is a fuzzy ideal of R.

Definition 1.14 [31]: Let Abe a fuzzy module of an R- module M and μ be a proper fuzzy submodule of A.Then μ is called a prime fuzzy submodule of a fuzzy module A if $r_t\, x_k \subseteq \mu$ for fuzzy singleton r_t of R and $x_k \subseteq A$, then either $r_t \subseteq [\mu:A]$ or $x_t \subseteq \mu$.

Definition 1.15 [28]: Let μ be anon-empty fuzzy submodule of a fuzzy module A of an R- module M.Annihilator of μ denoted by F-ann $\mu$is defined by:(F-an μ)(r)=sup$\{t : t \in [0,1], r_t\, \mu \subseteq O_1\}$,for all $r \in R$.
Note: F-ann $\mu = [O_1 : \mu]$, hence $(\text{F-ann } A)_t \subseteq \text{ann } A_t$.

Proposition 1.3 [4]: IfA is a fuzzy module of an R- module M, then F-ann A is a fuzzy ideal of R.

Definition 1.16 [9]: A fuzzy submoduleμ of a fuzzy module A is called a primary fuzzy submodule if for each $r_t\, x_k \in \mu$, then either $x_k \in \mu$ or $r_t^{\,n} \in [\mu:A]$ for some $n \in Z^+$ where r_t is a fuzzy singleton of R and $x_k \subseteq A$.

Proposition 1.4 [31]: If R is an R-module and A is afuzzy module of R-module R.Then I is a prime fuzzy ideal in R if and only if I is a prime fuzzy submodule of a fuzzy module A of R-module R.

Definition 1.17 [30]: A fuzzy submoduleμ of a fuzzy module A of an R- module M is said to be maximal fuzzy submodule if for any submodule σ of A, $\mu \subseteq \sigma$ either $\sigma = A$ or $\mu_* = \sigma_*$.

Definition 1.18 [30]: A fuzzy submoduleμ of a fuzzy module A of an R- module M is maximal fuzzy submodule if and only if im $(\mu) = \{1,t\}$, $t \ne 1$ and μ_* is a maximal submodule of M.

Theorem1.2 [24]: Let M be a finitely generated R- module M and let μ be a fuzzy submodule of a fuzzy module A. If θ is a fuzzy ideal of R such that $\theta\mu = \mu$ then there existst $\in [0,1]$ such that $\theta_t\, \mu = A$.

Theorem1.3 [27]: Let M be a finitely generated R- module M. Let θ be a fuzzy ideal of R and let μ be a fuzzy submodule of a fuzzy module A of R- module M. If $\theta\mu = \mu$ and $1 \in \text{im}(\mu)$ then $1 \in \text{im}(\theta)$ and $\mu(a) = 1$ for all $a \in M$.

Proposition 1.5 [31]: If $\mu$is a prime fuzzy submodule of a fuzzy module A of an R-module M, then $[\mu:A]$ is a prime.
The converse of the proposition is not true in general for example:
Let $M = Z \oplus Z$ as a Z-module, let$A : M \to [0,1]$, $\mu : M \to [0,1]$ defined by:

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$$A(a,b) = \begin{cases} 1 & \text{if } (a,b) \in 2Z \oplus Z \\ 0 & \text{otherwise} \end{cases}$$

$$\mu(a,b) = \begin{cases} 1 & \text{if } (a,b) \in 4Z \oplus \langle 0 \rangle \\ 0 & \text{otherwise} \end{cases}$$

It is clear that A is a fuzzy module of a Z-module M and μ is a fuzzy submodule of a Z-module M.Since $2_{\frac{1}{2}}(2,0)_{\frac{1}{2}} = (4,0)_{\frac{1}{2}} \subseteq \mu$ and $(2,0)_{\frac{1}{2}} \nsubseteq \mu$ but $2_{\frac{1}{2}} \nsubseteq [\mu:A]$, thin μ is not a prime fuzzy submodule. Since $[\mu_t:A_t]$ = [4Z⊕0, 2Z⊕Z]={0}, for all t>0.Hence $[\mu:A]_t$={0}, for all t>0, and $[\mu:A](r) = \begin{cases} 1 & \text{if } r = 0 \\ 0 & \text{otherwise} \end{cases} = 0_1$ which can be easily shown a prime fuzzy ideal of Z.

Proposition 1.6 [30]: If$\{\mu_i\}_{i\in I}$ a collection of fuzzy submodules of afuzzy module A of an R-module M then in $f_{i\in I}$ $[\mu_i:A]=[\inf_{i\in I} \mu_i:A]$.

II. Main Results

Theorem 2.1: Let μ, λ be two fuzzy submodules of a fuzzy module A of a module M and θ prime fuzzy submodule of afuzzy module A such that μ⋂λ⊆θ then either λ⊆θ or [μ:A]⊆[θ:A].

Proof: Suppose [μ:A]⊈[θ:A] then ∃ t∈[0,1] such that r_t⊆[μ:A] and r_t⊈ [θ:A]. Let s_k⊆λ, hences$_k$ r_t⊆μ⋂λ, k∈[0,1] therefor s_k r_t⊆θ. but θ is a prime fuzzy submodule and r_t⊈[θ:A] and hence s_k⊆θ this means λ⊆θ.

Corollary2.1: Let α be a fuzzy ideal of a ring R and Let μ be a fuzzy submodule of afuzzy module A of an R-module M. If θ prime fuzzy submodule of afuzzy module A such that μ⋂αA⊆θ. Then either αA⊆θ or μ⊆θ.

Proof: By theorem 2.1 we get μ⊆θ or [αA:A] ⊆ [θ:A]. If [αA:A]⊆[θ:A] then α⊆[θ:A] and hence αA⊆θ.

Theorem 2.2: Let $\{\mu_i\}_{i\in I}$ be a collection of prime fuzzy submodules of afuzzy module A of an R-module M,and let α a prime fuzzy ideal of a ring R. If α=$\inf_{i\in I}$ $[\mu_i:A]$, then $\inf_{i\in I} \mu_i$ is a prime fuzzy submodule.

Proof: Since α=$\inf_{i\in I}$ $[\mu_i:A]$ by proposition 1.6 α = $[\inf_{i\in I} \mu_i:A]$. Let r_t⊆ R, x_k⊆A such that r_t $x_k \subseteq \bigcap_{i\in I} \mu_i = \inf_{i\in I} \mu_i$. If $x_k \nsubseteq \bigcap_{i\in I} \mu_i$, then there is j ∈ I such that $x_k \nsubseteq \mu_j$. But r_t $x_k \subseteq \mu_j$ and μ_j Prime fuzzy submodule, hence $r_t \subseteq [\bigcap_{i\in I} \mu_i:A] = [\inf_{i\in I} \mu_i: A]$, this means $\inf_{i\in I} \mu_i$ is a prime fuzzy submodule.

Theorem2.3: If μ is a prime fuzzy submodule of a fuzzy module A of an R-module and λ is a fuzzy submodule of A such that λ⊄μ, then the fuzzy submodule μ⋂λ is prime in λ.

Proof: It is knownthat a proper fuzzy submodule μ⋂λ is a proper in λ. Let r_t⊆ R, x_k⊆ λ such that r_t x_k⊆ μ⋂λ. If x_k⊈ μ⋂λ then x_k⊈μ. But μ is prime fuzzy submodule and therefore r_t⊆ [μ:A] this means r_t A⊆μ and hence r_t λ ⊆ μ⋂λ this implies r_t⊆ [μ⋂λ:λ].

Theorem2.4: If μ a fuzzy maximal submodule of a fuzzy module A of an R-module M,then the fuzzy ideal [μ:A] is a maximal in R.

Proof: Since μ is a proper fuzzy submodule of a fuzzy module Aof an R-module M, then [μ:A] is a proper fuzzy ideal in R. Let η be a fuzzy ideal in R such that [μ:A]⊊η this means there exist y_l⊆η where y∈R, l∈[0,1] such that y_l ⊈ [μ:A], therefore exist m∈M and n∈[0,1] such that m_n y_l⊈μ. But μis a maximal and hence μ+⟨m_n y_l ⟩ = A ,where fuzzy submodule generating by ⟨m_n y_l⟩ for that there exist r∈R, x_k⊆ μ and k, t∈[0,1] here x_k+ r_t y_l m_n = m_n and this implies m_n (1_1-r_t y_l)⊆μ. We claim that 1_1-r_t y_l⊆ [μ:A]. To prove that , we will be suppose w_p⊆A, where w∈M and p∈[0,1], so that there exist s∈R, z_f⊆μ and q, f∈[0,1] such that z_f+s_q y_l m_n= w_p we observe (1_1-r_t y_l) w_p= (1_1- r_t y_l) s_q y_l m_n+ (1_1-r_t y_l)z_f⊆μ this means 1_1-r_t y_l⊆[μ:A]. But [μ:A]⊊η then 1_1-r_t y_l⊆η and hence we get 1_1⊆η and therefore η = 1_R, this implies [μ:A] is a maximal fuzzy ideal in R.

Theorem2.5: Let μ be a fuzzy submodule of a fuzzy module A of an R-module M.If the fuzzy ideal [μ:A] is a maximal in R ,then μ is a fuzzy prime submodule of a fuzzy module A of an R-module M.

Proof: Since [μ:A] is a proper fuzzy ideal in R,then μ is a proper fuzzy submodule of a fuzzy module A.Let r∈R, x∈M and t, k∈[0,1] such that r_t x_k⊆μ, if r_t⊈ [μ:A] then [μ:A]+⟨r_t⟩=1_R, and hence there exist s∈R and m_n⊆[μ:A]and m_n+s_l r_t=1_1 and therefore x_k=x_k m_n+x_k s_l r_t∈ μ this implies μ is a prime fuzzy submodule of a fuzzy module A.

Corollary2.2: If μ is a proper fuzzy submodule of a fuzzy module A of an R-module Mcontains a fuzzy submoduleλ such that [λ:A]is a fuzzy maximal ideal ,thenμ is a fuzzy prime submodule of A.

Proof: Sinceμ⊆λ ,then [μ:A]⊆[λ:A].But [λ:A] is a proper fuzzy ideal in R because λ is a proper fuzzy submodule of a fuzzy module A. Since [λ:A] is a maximal fuzzy ideal in R, then [μ:A]=[λ:A] and by Theorem 2.5μ is a fuzzy prime submodule of A.

Corollary 2.3: LetA be a fuzzy module of an R-module M and let α be a maximal fuzzy ideal in R. If A≠αA,then αA is a prime.

Proof: We claim that α=[αA:A],it is clear that α⊆[αA:A]. Now let r_t⊆ [αA:A], r∈R, t∈[0,1]. If r_t⊈α, then α + ⟨r_t ⟩ = 1_R and therefore there exist u, s∈R such that u_z+s_l r_t=1_1. ∀ m∈M, implies u_z m_n+s_l r_t m_n= m_n where z, n, l, n, t∈[0,1], and hence we observe that m_n ⊆ α A and therefore αA=A C!. This means r_t⊆ α and hence α = [αA:A] and by Theorem 2.5 the fuzzy submoduleαA is a prime fuzzy submodule of a fuzzy module A.

Corollary2.4: Every maximal fuzzy submodule is a prime fuzzy submodule.

Proof: It is clear.

The next proposition shows the condition to become the converse is true.

Proposition2.1: If μ is a primary fuzzy submoduleof a fuzzy module A of an R-module M,thenμ is a prime fuzzy submodule of A if and only if the fuzzy ideal [μ:A] is a prime in R.

Proof: Suppose thatμ is a primary fuzzy submodule of a fuzzy module A of an R-module M ,and [μ:A] is a prime fuzzy ideal. Let r∈R, x∈M and t, k ∈[0,1] such that r_t x_k⊆μ and suppose x_k ⊈ μ. Since μ is a primary fuzzy submodule of A, then there exist an integer positive number n such that r_t^n⊆ [μ:A], since [μ:A] is a prime fuzzy ideal, then r_t∈ [μ:A] and this implies that μ is a prime

fuzzy submodle of a fuzzy module A of an R-module M.
The converse from Proposition 1.5.

Corollary 2.5: If λ is a primary fuzzy submodule of a fuzzy module A of an R-module M and contains prime submodule μ such that $[\mu:A]=\sqrt{([\lambda:A])}$, then λ is prime fuzzy submodule of A.

Proof: It is clear that $[\mu:A]\subseteq[\lambda:A]\subseteq\sqrt{([\lambda:A])}=[\mu:A]$ and hence $[\mu:A]=[\lambda:A]$ this means $[\mu:A]$ is a prime fuzzy ideal inR.Since λ is a primary fuzzy submodule of a fuzzy module A of an R-module M by Proposition 2.1 λ is a prime fuzzy submodule of a fuzzy module A of an R-module M.
We need here definition before we give another case for the fuzzy submoduleμ to become prime when we suppose the fuzzy ideal $[\mu:A]$ is prime in R.

Definition2.1: Let A be a fuzzy module of an R-module M and α be a fuzzy ideal in R , a fuzzy submodule μ is said to be maximal fuzzy submodule from the type-α if the following holds:
1. $\alpha = [\mu:A]$
2. The fuzzy submodule μ is the maximam element in set of fuzzy submodule λ such that $\alpha = [\lambda:A]$

Note that: Every maximal fuzzy submodule μ of a fuzzy module A is a maximal fuzzy from the type $[\mu:A]$.

Theorem 2.6: Let μ be a fuzzy submodule of a fuzzy module A of an R-module M, and let α a fuzzy ideal in R such that μ maximal from the type $-\alpha$. The Fuzzy submodule μ is prime fuzzy submodule of a fuzzy module A if and only if the fuzzy ideal α is prime fuzzy ideal in R.

Proof: Let α be a prime fuzzy ideal in R. Since $\alpha=[\mu:A]$ is a proper prime fuzzy ideal in R,then μ is a proper fuzzy submodule of A. Now, let $r\in R$, $x\in M$ and $t, k\in[0,1]$ such that $r_t\, x_k\subseteq\mu$, suppose $x_k \not\subseteq\mu$. If we look the fuzzy submoduleλ such that $\mu+\langle x_k\rangle=\lambda$, it is clear that $\mu\subsetneq\lambda$, since the fuzzy submoduleμ is a maximal from type-α, then $\alpha \subsetneq [\lambda:A]$ and hence there exist $s \in R$, $l\in[0,1]$ and $s_l \not\subseteq \alpha$. In another hand we observe that for all $y \in M$ implies $s_l\, y_n \subseteq \lambda$ where $n \in [0,1]$ and hence there exist $m_p \subseteq \mu$, $v\in R$ and $p, d \in [0,1]$ such that $m_p+v_d\, x_k= s_l\, y_n$ .This implies $r_t\, m_p+r_t\, v_d\, x_k= r_t\, s_l\, y_n\subseteq\mu$, this means $r_t\, s_l\subseteq [\mu:A] = \alpha$. But α is a prime fuzzy ideal and $s_l\not\subseteq \alpha$, so $r_t \subseteq \alpha$ and therefore μ is a prime fuzzy submodule of A.
The converse is clear from Proposition 1.5.

Theorem 2.7: Letμ be a proper fuzzy submodule of a fuzzy module A of an R-module M,such that $[\lambda:A]\not\subseteq[\mu:A]$ for all fuzzy submodule λ of a fuzzy module A contains μ proporly. The fuzzy submodule$\mu$is prime fuzzy submodule if and only if the fuzzy ideal $[\mu:A]$ is prime fuzzy ideal in R.

Proof: Suppose that the fuzzy ideal $[\mu:A]$ is a prime in R, we want to prove that the fuzzy submodule μ is prime in A. Let $r\in R$, $x_k \subseteq A$ and $t, k \in [0,1]$ such that $r_t\, x_k \subseteq\mu$, suppose $x_k \not\subseteq \mu$ it is clear that the fuzzy submodule$\mu+\langle x_k\rangle = \lambda$ contains μ properly, so that $[\lambda:A] \subsetneq [\mu:A]$ this implies that there exist $s_l\subseteq [\lambda:A]$ and $s_l \not\subseteq [\mu:A]$ and hence $s_l\, A \subseteq \lambda$ and $s_l\, A \not\subseteq \mu$. But $r_t\, s_l \subseteq \mu$, this means $r_t\, s_l \subseteq [\mu:A]$, since $[\mu:A]$ is a prime fuzzy sub module and $s_l\not\subseteq [\mu:A]$, and therefore $r_t \subseteq [\mu:A]$, and hence we proved that μ is a prime fuzzysubmodule in A.

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