The present paper provides the steady state analysis of queueing system with phase type service in the sense that one service channel is linked in series with two non serial service channels which are in biseries. The input process is Poisson and service time distribution is exponential and service discipline follows FCFS (First come first served). In this, the customer may leave the system without getting the service. Waiting space is infinite in model-A and finite in model-B. For this particular queueing system, formulas for mean queue length, steady state marginal probabilities have been derived. Numerical solutions have been discussed and conclusions have also been derived from possible outcomes.

Keywords
Serial Queueing Process, Phase Type Service, Reneging, Numerical Solutions

I. Introduction
O’Brien [8], Jackson [4] and Hunt [3] studied the problems of serial queues in the steady state with Poisson assumptions. In these studies, it is assumed that a unit is not allowed to renege at any stage and must go through each service channel before leaving the system. Singh, Man [9] considered the Steady-state behavior of serial queueing processes with impatient customers. The queueing model having multiple parallel channels in series with impatient customers are obtained by Singh, Man and Ahuja, Asha [10]. Singh, Man; Punam and Ashok Kumar [12] obtained Steady-state solutions of serial and non-serial queueing processes with reneging and balking due to long queue and some urgent message and feedback. However, this paper provides the steady-state analysis of a first come first served queueing process where a unit may renege the system at any stage and each service channel has an input both from within and outside the system. Further, to make the system more practical, the concepts of queues in biseries is also included.

This paper deals with a queueing process where a service channel is linked in series with two non-serial service channels which are in biseries. Units demanding different types of service arrive at three service channels $S_1$, $S_2$ and $S_3$, after completion of service at $S_1$, they may leave the system or join either of service channels $S_2$ and $S_3$ for next phase of service. Waiting space is infinite in model-A and finite in case of model-B. The expression for the mean queue lengths has also been derived for model A.

II. Formulation of Model
The problem considered may be stated as follows: The system consists of three queues $Q_1$, $Q_2$ and $Q_3$ with respective servers $S_1$, $S_2$ and $S_3$. The input process is Poisson. Units arrive from outside the system in Poisson streams with parameters $\lambda_1$, $\lambda_2$ and $\lambda_3$ at $S_1$, $S_2$ and $S_3$ respectively. Further the units joining $Q_1$, $Q_2$, $Q_3$ may leave the system at random with mean reneging rates $\alpha_1$, $\alpha_2$ and $\alpha_3$ respectively. After the completion of the service at $S_1$, units leave the system with probability $q_1$ or join either of the queues $Q_2$ and $Q_3$ for the next phase of the service, with respective probabilities $q_2$ and $q_3$ where $q_1+q_2+q_3=1$. Again units serviced by $S_2$ either leave the system or join the queues $Q_2$ with respective probabilities $r_1$ and $r_2$ such that $r_1+r_2=1$. Similarly units serviced by $S_3$ either leave the system or join the queue $Q_3$ with respective probabilities $p_1$ and $p_2$ where $p_1+p_2=1$. The service time distributions for the servers $S_2$, $S_3$ and $S_3$ are represented by parameters $\mu_2$, $\mu_3$ and $\mu_3$ and are mutually independent negative exponential distribution.

The practical situations where such a model finds application are of common occurrence. Consider a situation in barber shop where the common counter $(S_1)$ is linked separately with two service facilities consisting of a hairdo service $(S_2)$ and grooming service $(S_3)$. The common counter provides service facilities such as appointment, providing information about offers going on etc. to the customers. It is a common practice that customer may go through all service centers i.e. common counter, hair cut and grooming. It is also not usual that the customer may leave the service facility unserved due to some reasons. Moreover, when the appointment etc. is not essential, the customers can join either of the two service facilities $(S_2, S_3)$ directly.

Model-A

III. FORMULATION OF EQUATION
A. Define:
$P(m|n|r|t)$= the probability that at time $t$ there are $m$ units (which may renege, or after being serviced by $S_1$, leave the system or join either of $S_2$ and $S_3$) waiting in $Q_1$; $n$ units (which may renege, or after being serviced by $S_2$, either leave the system or join $S_3$) waiting in $Q_2$; $r$ units (which may renege, or after being serviced by $S_3$, either leave the system or join $S_3$) waiting in $Q_3$.

Elementary probabilities reasoning gives the following difference-differential equations:

$$\frac{\partial}{\partial t}P(m|n|r|t) = -\{\lambda_1 + \lambda_2 + \lambda_3 + \delta(n)(\mu_1 + \alpha_2) + \delta(n)(\mu_2 + \alpha_3)\}P(m|n|r|t) + \lambda_1P(m-1|n|r|t) + \lambda_2P(m-1|n-1|r-1|t) + \lambda_3P(m-1|n-1|r-1|t) + (\mu_1\mu_2)P(m+1|n+1|r+1|t) + (\mu_2\mu_3)P(m+1|n+1|r+1|t) + (\mu_3\mu_3)P(m+1|n-1|r+1|t) + (\mu_2\mu_3)P(m-1|n-1|r+1|t) + (\mu_3\mu_3)P(m|n+1|r+1|t) + (\mu_3\mu_3)P(m|n-1|r+1|t)$$

(1)

where $m, n, r \geq 0$

$$\delta(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

and, $P(m|n|r|t) = 0$, if any of arguments are negative.

IV. Steady State Solution
The steady-state equations of the system are obtained by putting the time derivatives equal to zero in the equation (1)

The solutions of the resulting equations can be verified to be

$$P(m|n|r) = p_1^mp_2^n\rho_3^rP(0|0|0)$$

(2)

Where
\[ \rho_1 = \frac{\lambda_1}{\mu_1 + \lambda_1} \]
\[ \rho_2 = \frac{(\mu_2 + \alpha_3)(\lambda_2 + \mu_1 q_2 \rho_1) + p_2 \mu_3 (\lambda_3 + \mu_1 q_3 \rho_1)}{(\mu_2 + \alpha_2)(\mu_3 + \alpha_3) - r_2 \mu_2 p_2 \mu_3} \]
\[ \rho_3 = \frac{(\mu_2 + \alpha_2)(\lambda_3 + \mu_1 q_3 \rho_1) + r_2 \mu_2 (\lambda_2 + \mu_1 q_2 \rho_1)}{(\mu_2 + \alpha_2)(\mu_3 + \alpha_3) - r_2 \mu_2 p_2 \mu_3} \]

And \( P(0 \mid 0 \mid 0) \) can be determined from normalized conditions

\[ \sum_{m,n,r=0}^{\infty} P(m|n|r) = 1 \]  
(3)

Equations (2) and (3) give

\[ P(0 \mid 0 \mid 0) = (1 - \rho_1)(1 - \rho_2)(1 - \rho_3) \]

With the conditions \( \rho_1, \rho_2, \rho_3 < 1 \)

Thus

\[ P(m|n|r) = (1 - \rho_1)(1 - \rho_2) \rho_1^m \rho_2^n \rho_3^r ; m,n,r \geq 0 \]

(4)

V. Steady-State Marginal Probabilities

Let \( P(m) \) is the steady-state marginal probability that there are \( m \) units in the queue \( Q_1 \). This is determined as

\[ P(m) = \sum_{n=0}^{\infty} P(m|n|r) \]

Which gives:

\[ P(m) = \rho_1^m (1 - \rho_1) \]

(5)

Similarly \( P(n) \) and \( P(r) \) are given by

\[ P(n) = \rho_2^n (1 - \rho_2) \]

\[ P(r) = \rho_3^r (1 - \rho_3) \]

VI. Mean Queue Length

The mean queue length \( L_1 \) before the server \( S_1 \) is determined by using (4) in the formula

\[ L_1 = \sum_{m=0}^{\infty} m P(m) = \frac{\rho_1}{1 - \rho_1} \]

Similarly \( L_2 \) and \( L_3 \) are given by

\[ L_2 = \frac{\rho_2}{1 - \rho_2} \]

\[ L_3 = \frac{\rho_3}{1 - \rho_3} \]

Thus \( L \) the mean queue length of the system is

\[ L = L_1 + L_2 + L_3 = \frac{\rho_1}{1 - \rho_1} + \frac{\rho_2}{1 - \rho_2} + \frac{\rho_3}{1 - \rho_3} \]

(6)

Particular Cases

If \( \lambda_1 = \lambda_2 = \lambda_3 = 0 \); \( \alpha_1 = \alpha_2 = \alpha_3 = 0 \); \( r_1 = 1 = q_3 \); \( p_1 = p_2 = p_3 \) then

\[ P(m|n|0) = (1 - \rho_1)(1 - \rho_2)(1 - \rho_3) \rho_1^m \rho_2^n \rho_3^0 \]

\[ L = \frac{\rho_1}{1 - \rho_1} + \frac{\rho_2}{1 - \rho_2} + \frac{\rho_3}{1 - \rho_3} \]

(7)

Where

\[ \rho_1 = \frac{\lambda_1}{\mu_1 + \lambda_1} \]
\[ \rho_2 = \frac{\lambda_2 + \mu_1 q_2 \rho_1}{\mu_2 + \lambda_2 + \mu_1 q_2 \rho_1} \]
\[ \rho_3 = \frac{\lambda_3 + \mu_1 q_3 \rho_1}{\mu_2 + \lambda_3 + \mu_1 q_3 \rho_1} \]

With the conditions \( \rho_1, \rho_2, \rho_3 < 1 \)

The results in (6) are in agreement with the corresponding results of Jackson [4].

Model-B

VII. Formulation of Equations

Define \( P(m \mid n \mid r) \) as in model-A

we assume that if at any instant there are \( K \) units in the system, i.e. \( m+n+r=K \) then an arriving unit will not be allowed to join any of the queues and is considered lost for the system.

The following difference differential equations hold:

\[ \frac{d}{dt} P(m|n|r|t) = -[\delta(m)(\mu_1 + \alpha_1) + \delta(n)(\mu_2 + \alpha_2) + \delta(r)(\mu_3 + \alpha_3)] P(m|n|r|t) + \lambda_1 P(m+1|n|r|t) + \lambda_2 P(m|n+1|r|t) + \lambda_3 P(m|n|r+1|t) + (\rho_1 q_2) P(m+1|n+1|r+1|t) + (\rho_2 q_3) P(m+1|n+1|r+1|t) + (\rho_3 q_1) P(m+1|n+1|r+1|t) + (p_1 p_2) P(m|n+1|r+1|t) + (p_2 p_3) P(m|n+1|r+1|t) + (p_3 p_1) P(m|n+1|r+1|t) \]

(7)

\[ \frac{d}{dt} P(m|n|r|t) = -[\delta(m)(\mu_1 + \alpha_1) + \delta(n)(\mu_2 + \alpha_2) + \delta(r)(\mu_3 + \alpha_3)] P(m|n|r|t) + \lambda_1 P(m|n+1|r+1|t) + \lambda_2 P(m|n+1|r+1|t) + \lambda_3 P(m|n+1|r+1|t) + (\rho_1 q_2) P(m+1|n+1|r+1|t) + (\rho_2 q_3) P(m+1|n+1|r+1|t) + (\rho_3 q_1) P(m+1|n+1|r+1|t) + (p_1 p_2) P(m|n+1|r+1|t) + (p_2 p_3) P(m|n+1|r+1|t) + (p_3 p_1) P(m|n+1|r+1|t) \]

(8)

Where \( m, n, r \geq 0 \); \( m + n + r < K \)

VIII. Steady-State Solution

The steady state equations of the system are obtained by equating the time-derivatives to zero in the equations (7) and (8). The solution of the resulting can be verified to be

\[ P(m|n|r) = \rho_1^m \rho_2^n \rho_3^r P(0|0|0) \]

where \( m, n, r \geq 0 \); \( m + n + r \leq K \)

(9)

Where \( \rho_1, \rho_2, \rho_3 \) are defined as in model-A

P(0 | 0 | 0) can be determined from the normalized conditions

\[ \sum_{m,n,r=0}^{K} P(m|n|r) = 1 \]

(10)

With relation \( m + n + r \leq K \).

\[ [P(0|0|0)]^{-1} = \frac{1 - \rho_3^{K+1}}{(1 - \rho_1)(1 - \rho_2)(1 - \rho_3)} + \frac{\rho_1^2 (\rho_2^{K+1} - \rho_3^{K+1})}{(1 - \rho_1)(1 - \rho_2)(1 - \rho_3)} \]

Thus \( P(m | n | r) \) is determined completely

IX. Steady State Marginal Probabilities

Let \( P(m) \) is the steady-state marginal probability that there are \( m \) units in the queue \( Q_2 \). This is determined as

\[ P(m) = \sum_{n=0}^{m} P(m|n|r) \]

\[ n + r \leq K - m \]

Which gives:

\[ P(m) = P(0|0|0) \rho_1^m \left( (p_2 p_3)(1-p_1)(1-p_2)^{K+1} - (p_1 p_2)(1-p_3)^{K+1} \right) \]

(11)

Similarly \( P(n) \) and \( P(r) \) are expressed as

\[ P(n) = P(0|0|0) \rho_2^n \left( (p_1 p_3)(1-p_2)(1-p_3)^{K+1} - (p_2 p_3)(1-p_1)^{K+1} \right) \]

\[ P(r) = P(0|0|0) \rho_3^r \left( (p_1 p_2)(1-p_3)(1-p_3)^{K+1} - (p_3 p_1)(1-p_2)^{K+1} \right) \]
X. Mean Queue Length

The mean queue length, \( L_1 \), before the server \( S_1 \) is obtained by using (11) in the formula.

\[
L_1 = \sum_{n=0}^{K} mP(m) = \frac{P(0|0)}{(\rho_1 - \rho_2)(1 - p_3)} \left( \frac{\rho_2(1 + (K + 1)\rho_1^K + K\rho_2^K)}{(1 - \rho_1)^2} \right) + \left( \frac{\rho_1^2\rho_2^2(1 - (K + 1)\rho_2^K)(\rho_1 - \rho_3)}{(1 - \rho_2)^2} \right)
\]

Similarly \( L_2 \) and \( L_3 \) are given by

\[
L_2 = \frac{P(0|0)}{(\rho_2 - \rho_3)(1 - p_1)} \left( \frac{\rho_1(1 + (K + 1)\rho_2^K + K\rho_2^K)}{(1 - \rho_2)^2} \right) + \left( \frac{\rho_2(1 - (K + 1)\rho_2^K)(\rho_2 - \rho_3)}{(1 - \rho_2)^2} \right)
\]

\[
L_3 = \frac{P(0|0)}{(\rho_2 - \rho_3)(1 - p_1)} \left( \frac{\rho_1(1 + (K + 1)\rho_2^K + K\rho_2^K)}{(1 - \rho_2)^2} \right) + \left( \frac{\rho_2(1 - (K + 1)\rho_2^K)(\rho_2 - \rho_3)}{(1 - \rho_2)^2} \right)
\]

Thus \( L \), the mean queue length of the system is given by

\[
L = L_1 + L_2 + L_3
\]

Particular case

If \( \lambda_1 = \lambda_2 = 0 = \lambda_3 = 0 = \mu_2 = 0 = q_2 = 1 = \mu_1 = 0 = p_1 = 0 = p_2 \), then

\[
P(m|n) = \frac{\rho_1^m \rho_2^n P(0|0)}{(\rho_1 - \rho_2)(1 - p_3)(1 - p_2)}
\]

Where \( \rho_1 = \frac{\lambda_1}{\mu_1}, \rho_2 = \frac{\lambda_2}{\mu_2} \)

\[
P(0|0) = \frac{(\rho_1 - \rho_2)(1 - p_3)(1 - p_2)}{(\rho_1 - \rho_2)(1 - p_3)(1 - p_2) + \rho_1 \rho_2 (\rho_1^2 + \rho_1 \rho_2 + \rho_2^2)}
\]

The mean queue length of the system in this case is

\[
L = \frac{\rho_1^2[1 - (K + 1)\rho_2^K + K\rho_2^K]}{(1 - \rho_2)^2} - \frac{\rho_2^2[1 - (K + 1)\rho_2^K + K\rho_2^K]}{(1 - \rho_2)^2}
\]

IX. Numerical Solution of Model A

<table>
<thead>
<tr>
<th>Arrival rate (per hour)</th>
<th>Service rate (per hour)</th>
<th>Reneging rate (per hour)</th>
<th>Probabilities</th>
<th>Probabilities</th>
<th>Probabilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda_i )</td>
<td>( \mu_i )</td>
<td>( a_i )</td>
<td>i=1,2</td>
<td>i=1,2</td>
<td>i=1,2</td>
</tr>
<tr>
<td>41</td>
<td>90</td>
<td>8</td>
<td>1/2</td>
<td>6/7</td>
<td>3/4</td>
</tr>
<tr>
<td>18</td>
<td>95</td>
<td>12</td>
<td>3/10</td>
<td>1/7</td>
<td>4/1</td>
</tr>
<tr>
<td>20</td>
<td>100</td>
<td>15</td>
<td>1/5</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[
\rho_1 = \frac{\lambda_1}{\mu_1 + \alpha_1} = \frac{41}{96} = 0.418
\]

\[
\rho_2 = \frac{(\mu_1 + \lambda_1)(\mu_2 + \lambda_2) + \alpha_2(\lambda_2 + \mu_2 \lambda_2 \mu_2)}{(\mu_2 + \lambda_2)(\mu_2 + \lambda_2) - \alpha_2 \mu_2 \lambda_2 \mu_2} = \frac{4055.99}{11965.714} = 0.339
\]

\[
\rho_3 = \frac{(\mu_3 + \lambda_3)(\mu_3 + \lambda_3 \mu_3) + \alpha_3(\lambda_3 + \mu_3 \lambda_3 \mu_3)}{(\mu_3 + \lambda_3)(\mu_3 + \lambda_3) - \alpha_3 \mu_3 \lambda_3 \mu_3} = \frac{3342521}{11965714} = 0.279
\]

Mean Queue Length

\[
L = L_1 + L_2 + L_3 = \frac{\rho_1}{1 - \rho_1} + \frac{\rho_2}{1 - \rho_2} + \frac{\rho_3}{1 - \rho_3} = 0.418 + 0.582 + 0.718 = 1.718
\]

\[
L_2 = \frac{0.399}{0.661} = 0.513
\]

\[
L_3 = \frac{0.279}{0.721} = 0.387
\]

\[
L = L_1 + L_2 + L_3 = 0.718 + 0.513 + 0.387 = 1.618
\]

XII. Conclusion

It can be seen that mean queue length of this particular queuing system is found to be 1.618≈2. This number is quite impressive for successful queuing model. This queue length is achieved by giving larger values to service rates as compared to arrival rates and also in this numerical solution reneging rates are quite high. Servers do not have any control over arrival rates and reneging rates of units arriving for service. To control mean queue length of system, service rates can be increased by servers. This will make system more well-organized and also it will help to reduce reneging rates.

References


Ms. Rajvinder Kaur is presently working as Assistant Professor in the Department of Mathematics at M.M. Modi College, Patiala, Punjab. She is M.Phil in Mathematics and her area of research is Queues and its applications. She has around five years of teaching experience.